Cellular Sheaves, Sheaf Laplacian, Harmonicity etc.

~~Cell complexes = Generalizes graphs~~

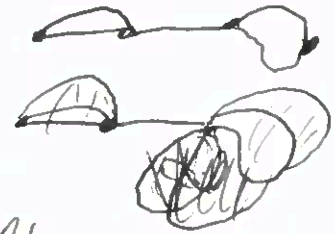
Prelims:

Cell complexes = Generalize graphs.

informally: Top. spaces obtained by attaching discs along their boundary.

Base level: points

1-cells: Attach to 0-cells



2-cells: " to 1-skeleton

3-cells solid

→ solid. solid.

Def^n: A (finite) regular cell complex $X \in Top$, has a (finite) partition into subspaces $\{X_\alpha\}_{\alpha \in P_X}$ $(X = \sqcup X_\alpha)$

(0) $X_\alpha \cong \mathbb{R}^{n_\alpha}$ some n_α (interior of cells)

(1) $\overline{X}_\alpha \cap X_\beta \neq \emptyset \Rightarrow X_\beta \subseteq \overline{X}_\alpha$ (write $\beta \trianglelefteq \alpha$) "$\beta$ is a face of α"

(2) $\exists$ homeo. $\mathbb{D}^n \to \overline{X}_\alpha$ s.t. $\mathbb{R}^{n_\alpha} \xrightarrow{\sim} X_\alpha$ Extends to $\mathbb{R}^{n_\alpha} \subset \mathbb{D}^{n_\alpha} \to \overline{X}_\alpha$

Notice $P_X = \{\text{cells}\}$ is a poset $(\beta \trianglelefteq \alpha)$

Morphisms $X \to Y$: a cont. map lifting a map of posets $P_X \to P_Y$ s.t. $n_{\varphi(\alpha)} \leq n_\alpha$ (if $\varphi(\overline{\alpha}) = \overline{\varphi(\alpha)}$, then cts auto.)

Note: (2) excludes , . Its regularity ensures X can be reconstructed from P_X (see Hansen's thesis)

Def^n: k-skeleton $X^{(k)} = \bigsqcup_{n_\alpha \leq k} X_\alpha$,

$\sigma \in P_X \Rightarrow st(\sigma)$ = star of σ = $\{\tau \in P_X : \sigma \trianglelefteq \tau\}$

⇝ smallest open collection of cells cont σ.

~~Cellular Sheaves~~ : See Hansen & references for which posets arise from regular cell cxs

Props: Pr

Cellular Sheaves

A sheaf on top. spaces: $\forall U \subset Op(X) \mapsto \mathcal{F}(U)$, $\forall U \subseteq V \rightsquigarrow \mathcal{F}(U) \to \mathcal{F}(V)$ } sheaf cond.

<u>Def</u>: A <u>cellular sheaf</u> (of vector spaces) on a cell cx X is a covariant functor $\mathcal{F}: P_X \to \mathrm{Vect}_{\mathbb{R}}$

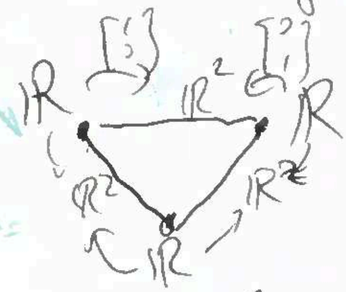
(1) $\forall \sigma \in P_X \mapsto \mathcal{F}(\sigma)$: vector space.

(2) $\forall \sigma \trianglelefteq \tau \mapsto \mathcal{F}_{\sigma \trianglelefteq \tau} : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ s.t.

$$\rho \trianglelefteq \sigma \trianglelefteq \tau \Rightarrow \mathcal{F}_{\rho \trianglelefteq \tau} = \mathcal{F}_{\sigma \trianglelefteq \tau} \circ \mathcal{F}_{\rho \trianglelefteq \sigma}$$

: $\mathcal{F}(\sigma)$ stalk of $\mathcal{F}$ at σ, $\mathcal{F}_{\sigma \trianglelefteq \tau}$: restr. maps

E.g.



<u>Ex</u>: const. shv. $\mathcal{F}_\sigma = V$, $\mathcal{F}_{\sigma \trianglelefteq \tau} = \mathrm{id}$

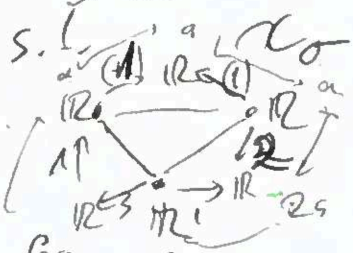
! Related to constructible sheaves (• Curry's thesis • Exit path cats (Lurie, Treumann))

<u>Defⁿ</u>: <u>Cosheaf</u>: reverse the direction of arrows

<u>Defⁿ</u>: Global section of $\mathcal{F}$: choice of $x_\sigma \in \mathcal{F}(\sigma)$ $\rightsquigarrow \Gamma(X; \mathcal{F})$

s.t. $x_\sigma \mapsto x_\tau$ $\forall \sigma \trianglelefteq \tau$

E.g.



$6a = a$. (May skip this ex.)

Sheaf morphism: $\mathcal{F}, \mathcal{G}$ on X, Morph. $\equiv$ Nat. trf $P_X \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightrightarrows}} \mathrm{Vect}$

Direct sum: $(\mathcal{F}\oplus\mathcal{G})(\sigma) = \mathcal{F}(\sigma)\oplus\mathcal{G}(\sigma)$

Tensor prod: Obvious.

Pull-back: $X \xrightarrow{f} Y \Rightarrow f^*\mathcal{F}(\sigma) = \mathcal{F}(f(\sigma))$

Push-forward: $(f_*\mathcal{F})(\sigma) = \lim_{\sigma \trianglelefteq f(\tau)} \mathcal{F}(\tau)$ $\left(0 \text{ if } \sigma \ntrianglelefteq f(\tau)\ \forall\tau\right)$

Cohomology: (cf Cech complex)

$$C^k(X;\mathcal{F}) = \bigoplus_{\dim(\sigma)=k}\mathcal{F}(\sigma)$$

Want: Chain complex $C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \cdots$, s.t. $\delta(x) = \sum_{\dim(\tau)=k+1} \pm \mathcal{F}_{\sigma\trianglelefteq\tau}(x)$

* $\delta^2=0 \iff$ ~~$\sigma\trianglelefteq\tau\trianglelefteq\eta$~~ $\sigma \overset{2}{\trianglelefteq} \eta \Rightarrow \left(\sum_{\dim\tau=\dim(\sigma)+1} \pm 1\right)\mathcal{F}_{\sigma\trianglelefteq\eta} = 0$

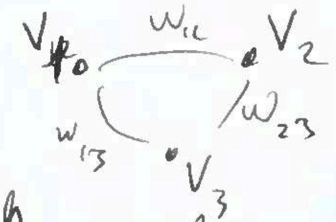
Signed incidence relation: $[\cdot,\cdot]: P_X\times P_X \to \{0,\pm1\}$

1) if $[\sigma:\tau]\neq 0$, then $\sigma\trianglelefteq\tau$, & no cells btw σ & τ in P_X

2) $\sum_{\gamma\in P_X}^{\sigma\trianglelefteq\tau} [\sigma:\gamma][\gamma:\tau]=0$. (Rk: $[\cdot,\cdot]$ think of the orientation of a face)

Def^n: $\delta^k(x) = \sum_{\dim\tau=k+1}[\sigma:\tau]\,\mathcal{F}_{\sigma\trianglelefteq\tau}(x)$, $x \in \mathcal{F}(\sigma)$

Example (Graphs)



$V_1\oplus V_2\oplus V_3 \to W_{12}\oplus W_{23}\oplus W_{13}$

Ex. Const. sheaf on graphs. $\mathbb{R}^{|V|} \longrightarrow \mathbb{R}^{|E|}$

$[\cdot,\cdot]$ = orient edges, $[v,e] = \begin{cases}1 & \text{if } v=s(e)\\ -1 & \text{if } v=t(e)\end{cases}$ $\qquad v \longmapsto -\sum_{\text{source}=v}\text{edges} + \sum_{\text{target}=v}\text{edges}$

~~2)~~ adjoint/dual to $e \overset{\partial}{\mapsto} t(e)-s(e)$ (cf. last week)

Exercise: $H^0(X;F) = \Gamma(X;F)$ (& is it?)

Relative version: $A \subset X$ subcomplex $\Rightarrow$

$$0 \to C^\bullet(X, A; F) \xrightarrow{\ker} C^\bullet(X;F) \to C^\bullet(A;F) \to 0$$

$\ni (x_\sigma)$ s.t. $x_\sigma = 0$ if $\sigma \in A$

$\rightsquigarrow 0 \to H^0(X, A; F) \to H^0(X;F) \to H^0(A;F) \to \dots$

Laplacian: $\boxed{\text{wtd sheaf}}$: functor $P_X \to$ Hilbert spaces

Recall graph Lapl. $C^1 = \mathbb{R}^{|E|}$, $C^0 = \mathbb{R}^{|V|}$, $C^1 \underset{\partial^T}{\overset{\partial}{\rightleftarrows}} C^0$

$L = \partial \circ \partial^T$

PDE Laplacian: X compact Riem. mfd. $\Rightarrow$ $\Omega^0_X \underset{\delta = d^*}{\overset{d}{\rightleftarrows}} \Omega^1_X \underset{\delta = d^*}{\overset{d}{\rightleftarrows}} \dots \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^k_X$

$\Delta = \delta \circ d + d \circ \delta$, $\Delta(\alpha) = 0 \Rightarrow$ Harmonic.

Fact. Harmonic forms compute coh. & $\Omega^i_X = \text{Harmonics} \oplus \text{im}(d) \oplus \text{im}(d^*)$

(Hodge decomposition)

More generally: consider a ch. complex of Hilbert spaces

$C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \dots$, let $\delta^* =$ adjoint of δ i.e.

$\langle \delta x, y \rangle = \langle x, \delta^* y \rangle$. $\quad ((\delta^*)^2 = \delta^2 = 0)$

Laplacian: $\Delta = \underbrace{\delta^* \delta}_{\Delta_+} + \underbrace{\delta \delta^*}_{\Delta_-} = (\delta + \delta^*)^2$.

[$x \in C^i$ = Harmonic if $\Delta(x) = 0 \Rightarrow \mathcal{X}^k(C^\bullet)$

Note: $\Delta(x) = 0 \Leftrightarrow \delta x = \delta^* x = 0$

(prove bel)]

Thm Assume $C^0, C^1, \dots$ finite dim'l. $\ker(\Delta^k) = H^k(C^\bullet)$

& $C^k = \text{im}(\delta) \oplus \text{im}(\delta^*) \oplus \mathcal{X}^k(C^\bullet)$

Pf: $\Delta(x) = 0 \Rightarrow \delta^*\delta \Delta(x) = 0 \Rightarrow (\delta^*\delta)^2(x) = 0 \Rightarrow \langle \delta^*\delta(x), x \rangle = 0 \Rightarrow \delta^*\delta(x) = 0$

~~similarly $\delta^*\delta$~~ $\Rightarrow \langle \delta^* \delta x, x \rangle = 0 \Rightarrow \delta x = 0$

Similarly $\delta^*(x) = 0$

Harmonic

$x + \delta(y) + \delta^*(z) = 0 \Rightarrow$ check $x = \delta(y) = \delta^*(z) = 0$. If $x \in C^i$, $x - \delta\delta^*(x) - \delta^*\delta(x)$

C^k also $\ker(\delta) = \mathrm{im}(\delta^*)^\perp \supseteq \mathrm{im}(\delta)$. $\mathrm{im}(\delta)^\perp = \ker(\delta^*)$

$\mathrm{im}(\delta)^\perp \cap \ker(\delta) = \ker\delta \cap \ker\delta^* = \text{Harmonics}$

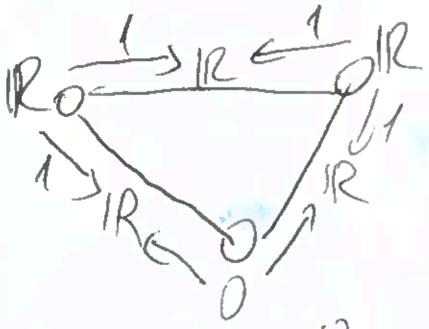
So $C^k = \mathrm{im}(\delta^*) \oplus \mathrm{im}(\delta) \oplus \text{Harmonics}$ □

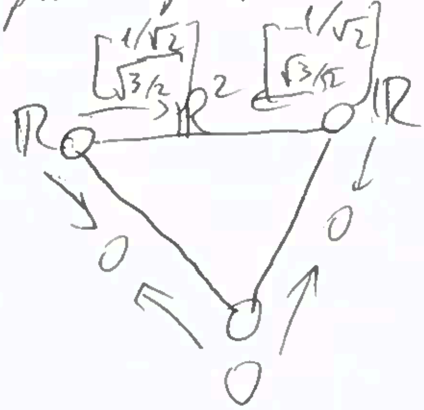
Def: $\mathcal{H}^k(C^\bullet) = \text{Harmonic cochains}$

Ex: Δ^0 for a graph is the graph Lapl. $\partial \circ \partial^T$ ($\delta\delta^*$ vanish for degree reasons)

Sheaf Lapl.: if each $\mathcal{F}(\sigma) \in$ Hilbert sp., $\Rightarrow \exists$ Laplacian on $C^*(X;\mathcal{F})$, $\Delta_{\mathcal{F}}$.

Example: wtd labeled graphs $\overset{1-1}{\longleftrightarrow}$ graph Lapl.





$$\mathbb{R}^2 \xrightarrow{\begin{bmatrix}1 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix}} \mathbb{R}^3 \qquad \mathbb{R}^2 \xrightarrow{\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ \sqrt{3}/2 & -\sqrt{3}/2\end{bmatrix}} \mathbb{R}^2$$

$$\Delta = \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \qquad \Delta = \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$$

Weighted Sheaves & Normalized Laplacian

norm. graph Lapl. $D^{-1/2} L D^{-1/2}$ ≡

For sheaves we modify wts.

Def: $\mathcal{F}$ wtd cellular sheaf on X, normalized if ~~∀x,y~~

$\forall\sigma\ \forall x,y \in \mathcal{F}(\sigma) \cap \ker(\delta)^\perp$, $\langle \delta x, \delta y\rangle = \langle x, y\rangle$

Lemma: X fin. dim. Can always reweight so $\mathcal{F}$ is normalized.

Pf $\bigoplus_{\dim(\sigma)=0} \mathcal{F}(\sigma) \to \bigoplus \cdots \to \bigoplus_{\dim(\sigma)=k-1} \mathcal{F}(\sigma) \to \bigoplus_{\dim(\sigma)=k} \mathcal{F}(\sigma)$

✓ $(\delta = 0)$

On $\mathcal{F}(\sigma)$ s.t. $\dim(\sigma)=k-1$, $\boxed{\dfrac{\ker\delta^{\perp}}{\ker\delta}} \Big\} \mathcal{F}(\sigma) \hookrightarrow C^k$

- Induce inner prod on $\ker(\delta)^{\perp}$ using δ } ortho. sum.
- Same inner prod. on $\ker(\delta)$,

Induct down $\square$

Harmonicity

Harmonic extensions of $\mathcal{F}$ cell sh./X, $B \subseteq X$ subcx & let $x|_B$ $x \in C^k(B;\mathcal{F})$ a cochain on B. Assume $H^k(X,B;\mathcal{F})=0$. Then $\exists!\ \tilde{x} \in C^k(X;\mathcal{F})$ cochain s.t. $\tilde{x}|_B = x$ on B and is harmonic on $S = X \setminus B$

Spectra of Sheaf Laplacians

Recall: $\tilde{L}$ has evals $\in [0,2]$

Prop: $\mathcal{F}$ = normalized sh. on a ~~k~~ simplicial cx X.
The evals of Δ^k_{+} ($\delta^*\delta$) = up Laplacian $L^k_{\mathcal{F}}$ are bdd above by $k+2$

Some Applications

1) Distributed consensus (cf ~~Graph diffusion~~

Prop $\mathcal{F}$ on X. ~~Then~~ dyn. sys $\dot{x} = -\Delta^k_{\mathcal{F}} x$, $x(0) = x_0$. Th.3

converges exp. to orth. proj. of x_0 onto $H^k(X;\mathcal{F})$

Pf: $\Delta^k_{\mathcal{F}}$ self adj. w/ evals ≥ 0 $\left(\begin{matrix}\langle \Delta^k_{\mathcal{F}} x, x\rangle \\ = \langle \delta x, \delta x\rangle + \langle \delta^* x, \delta^* x\rangle\end{matrix}\right)$

Diagonalize $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\quad x(t) = e^{-\Delta^k_{\mathcal{F}} \cdot t} x_0$.

Components $\lambda_i > 0 \xrightarrow[\text{exp. fast}]{e^{-\lambda_i t}} 0$, comp. $\lambda_i = 0$ $\left(\begin{matrix}\Delta = 0 \\ \text{comp.} \end{matrix}\right.$ stays the same $\square$

$k = 0 \Rightarrow$ Consensus can be reached on the nearest global section to an initial cond.

2) Opinion dynamics (Two weeks)

3) others.